

Lecture 9

This lecture provides an introduction to:

- **relations between sets**
- **representations of relations**
- **special relations**

You should master the material contained in this lecture before moving on to the next lecture.

Part 5: Relations between Sets

Relations

Let X and Y be sets. We know that the Cartesian product of X and Y is the set of ordered pairs

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

A *relation* ρ between the sets X and Y (in that order) is defined to be any subset of $X \times Y$. So a relation is a set of ordered pairs of the form (x, y), where $x \in X$ and $y \in Y$.

Example:
Let $X = \{a, b, c, d\}$. Let $Y = \{1, 2, 3\}$. Then one relation between X and Y is

$$\rho = \{(a, 2), (c, 1), (c, 3)\}.$$

End of example

Note that there are many different relations between two sets. In fact, we can work out exactly how many there are just by knowing the number of elements in each of the two sets. This is because the relations between two sets X and Y are precisely the subsets of $X \times Y$, and we know how many subsets there are.

Suppose that X has m elements and Y

has n elements. Then their Cartesian product $X \times Y$ has mn elements. So the power set $\mathcal{P}(X \times Y)$ of $X \times Y$ has 2^{mn} elements. That is, $X \times Y$ has 2^{mn} subsets; and this must therefore be the number of relations between X and Y.

In summary, the number of relations between X and Y is $2^{(|X|\cdot|Y|)}$.

Example:
In the last example, $|X| = 4$ and $|Y| = 3$. So the number of relations between X and Y is:

$$2^{(4\cdot 3)} = 2^{12} = 4096$$

End of example

In Lecture 5 we remarked that functions can be regarded as special kinds of relations. Specifically, a function from a set X to a set Y is a relation ρ between X and Y such that every element of X is the first coordinate of exactly one ordered pair in ρ.

Since many relations do not have this property, the number of functions between two sets is usually less than the number of relations.

Example:
Consider the sets X and Y of the last two examples. Since $|X| = 4$ and $|Y| = 3$,

the number of functions from $\boldsymbol{X}$ to $\boldsymbol{Y}$ is:

$$|\boldsymbol{Y}|^{|\boldsymbol{X}|} = \mathbf{3^4 = 81}$$

So among the 4096 relations between $\boldsymbol{X}$ and $\boldsymbol{Y}$, only 81 are functions.

End of example

Worked Example:
Let $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}\}$ and let $\boldsymbol{Y} = \{\mathbf{1}, \mathbf{2}\}$. Write down all relations between $\boldsymbol{X}$ and $\boldsymbol{Y}$. Which of them are functions?

We begin by constructing the Cartesian product $\boldsymbol{X} \times \boldsymbol{Y}$. It is this set:

$$\{(\boldsymbol{a}, 1), (\boldsymbol{a}, 2), (\boldsymbol{b}, 1), (\boldsymbol{b}, 2)\}$$

Now every subset of this set is a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$; and, conversely, all relations between $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be obtained in this way.

The 0-element subset of $\boldsymbol{X} \times \boldsymbol{Y}$ is $\varnothing$.

The 1-element subsets of $\boldsymbol{X} \times \boldsymbol{Y}$ are:
$\{(\boldsymbol{a}, 1)\}, \{(\boldsymbol{a}, 2)\}, \{(\boldsymbol{b}, 1)\}, \{(\boldsymbol{b}, 2)\}$

The 2-element subsets of $\boldsymbol{X} \times \boldsymbol{Y}$ are:
$\{(\boldsymbol{a}, 1), (\boldsymbol{a}, 2)\}, \{(\boldsymbol{a}, 1), (\boldsymbol{b}, 1)\},$
$\{(\boldsymbol{a}, 1), (\boldsymbol{b}, 2)\}, \{(\boldsymbol{a}, 2), (\boldsymbol{b}, 1)\},$
$\{(\boldsymbol{a}, 2), (\boldsymbol{b}, 2)\}, \{(\boldsymbol{b}, 1), (\boldsymbol{b}, 2)\}$

The 3-element subsets of $\boldsymbol{X} \times \boldsymbol{Y}$ are:
$\{(\boldsymbol{a}, 1), (\boldsymbol{a}, 2), (\boldsymbol{b}, 1)\},$
$\{(\boldsymbol{a}, 1), (\boldsymbol{a}, 2), (\boldsymbol{b}, 2)\},$
$\{(\boldsymbol{a}, 1), (\boldsymbol{b}, 1), (\boldsymbol{b}, 2)\},$
$\{(\boldsymbol{a}, 2), (\boldsymbol{b}, 1), (\boldsymbol{b}, 2)\}$

The 4-element subset of $\boldsymbol{X} \times \boldsymbol{Y}$ is:

$$\boldsymbol{X} \times \boldsymbol{Y} = \{(\boldsymbol{a}, 1), (\boldsymbol{a}, 2), (\boldsymbol{b}, 1), (\boldsymbol{b}, 2)\}$$

The only functions among these 16 relations are the second, third, fourth and fifth of the six 2-element subsets listed above.

End of worked example

If $\boldsymbol{\rho}$ is a relation between sets $\boldsymbol{X}$ and $\boldsymbol{Y}$, and the ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ is in $\boldsymbol{\rho}$, then we also say that $\boldsymbol{x}$ *is related to* $\boldsymbol{y}$ (*by* $\boldsymbol{\rho}$) or that $\boldsymbol{x}$ *is* $\boldsymbol{\rho}$*-related to* $\boldsymbol{y}$, and we write

$$\boldsymbol{x \ \rho \ y.}$$

If $(\boldsymbol{x}, \boldsymbol{y})$ is not in $\boldsymbol{\rho}$ then $\boldsymbol{x}$ is **not** $\boldsymbol{\rho}$-related to $\boldsymbol{y}$, in which case we write

$$\boldsymbol{x \ \not\rho \ y.}$$

Example:
In the first example we had two sets $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ and $\boldsymbol{Y} = \{\boldsymbol{1}, \boldsymbol{2}, \boldsymbol{3}\}$ and a relation

$$\boldsymbol{\rho = \{(a, 2), (c, 1), (c, 3)\}.}$$

Since $(\boldsymbol{a}, \boldsymbol{2}) \in \boldsymbol{\rho}$ we can also write $\boldsymbol{a \ \rho \ 2}$ and say that $\boldsymbol{a}$ is $\boldsymbol{\rho}$-related to $\mathbf{2}$. Similarly, $\boldsymbol{c \ \rho \ 1}$ and $\boldsymbol{c \ \rho \ 3}$. But (for example) $\boldsymbol{a \ \not\rho \ 1}$, $\boldsymbol{b \ \not\rho \ 1}$, and so on.

End of example

Sometimes a relation between two sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ is defined by means of a mathematical relationship between some elements of $\boldsymbol{X}$ and some elements of $\boldsymbol{Y}$.

Worked Example:
Let $\boldsymbol{X} = \{\mathbf{1,2,3}\}$. Let $\boldsymbol{Y} = \{\mathbf{1,3,5,9,11}\}$. Consider the relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined by:

$$\boldsymbol{x}\ \boldsymbol{\rho}\ \boldsymbol{y} \text{ if } \mathbf{5} - \boldsymbol{x} > \boldsymbol{y}$$

We want to write down $\boldsymbol{\rho}$ as a set of ordered pairs. A systematic way is to look at the elements of $\boldsymbol{X}$, one at a time, and find the elements of $\boldsymbol{Y}$ to which that element of $\boldsymbol{X}$ is related.

If $\boldsymbol{x} = \mathbf{1}$, then $\mathbf{5} - \boldsymbol{x} = \mathbf{4}$ which is greater than the elements $\mathbf{1}$ and $\mathbf{3}$ of $\boldsymbol{Y}$ but no other elements of $\boldsymbol{Y}$. So $\mathbf{1}\ \boldsymbol{\rho}\ \mathbf{1}$ and $\mathbf{1}\ \boldsymbol{\rho}\ \mathbf{3}$.

If $\boldsymbol{x} = \mathbf{2}$, then $\mathbf{5} - \boldsymbol{x} = \mathbf{3}$ which is greater than the element $\mathbf{1}$ of $\boldsymbol{Y}$ but no other elements of $\boldsymbol{Y}$. So $\mathbf{2}\ \boldsymbol{\rho}\ \mathbf{1}$.

Similarly $\mathbf{3}\ \boldsymbol{\rho}\ \mathbf{1}$ but $\mathbf{3}$ is not $\boldsymbol{\rho}$-related to any other element of $\boldsymbol{Y}$.

So $\boldsymbol{\rho} = \{(\mathbf{1,1}), (\mathbf{1,3}), (\mathbf{2,1}), (\mathbf{3,1})\}$.

End of worked example

Exercise:
Suppose that we modify the last example, with $\boldsymbol{X}$ and $\boldsymbol{Y}$ being the same sets as before, and with $\boldsymbol{\sigma}$ being defined by:

$$\boldsymbol{x}\ \boldsymbol{\sigma}\ \boldsymbol{y} \text{ if } \mathbf{5} - \boldsymbol{x} < \boldsymbol{y}$$

Write down $\boldsymbol{\sigma}$ as a set of ordered pairs.

End of exercise

Representing Relations

There are several useful ways to represent relations. One is to use a *graphical representation.*

In Discrete Mathematics, *graphs* usually involve points (called *vertices*) and straight or curved line segments (called *edges*). An edge usually connects two vertices to each other. But sometimes a curved edge has both ends touching the one vertex, in which case it is called a *loop.*

A *directed graph* is very similar to a graph, except that between two vertices there may be an *arrow.* Arrows are also called *directed edges.*

A common way to graphically represent a relation $\boldsymbol{\rho}$ between sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ is to firstly construct two columns of vertices. In the left-hand column are vertices representing the elements of $\boldsymbol{X}$, while the right-hand column displays vertices representing the elements of $\boldsymbol{Y}$. The names of the elements in $\boldsymbol{X}$ and $\boldsymbol{Y}$ are written beside the corresponding vertices, as labels. There is an arrow from a vertex labelled $\boldsymbol{x}$ (where $\boldsymbol{x} \in \boldsymbol{X}$) to a vertex labelled $\boldsymbol{y}$ (where $\boldsymbol{y} \in \boldsymbol{Y}$) if and only if $\boldsymbol{x}\ \boldsymbol{\rho}\ \boldsymbol{y}$. So this representation uses a *directed graph* to display the relation.

Example:
Consider the relation $\boldsymbol{\rho}$ of the last example. It can be represented graphically as in Figure 1.

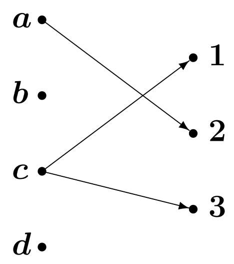

Figure 1: Graphical representation of a relation.

End of example

Another way to represent a relation is to use a *Boolean* or *logical matrix*.

A *matrix* is an array of numbers or elements, with the array enclosed in round or square brackets. The numbers or elements displayed in the matrix are the *entries* of the matrix. The entry in the ith row and the jth column is called the (i, j)-*entry*.

The logical matrix representation M_ρ of a relation ρ between sets X and Y has every entry being 0 or 1. There is a row for each element of the set X, and a column for each element of the set Y. The (x, y)-entry is 1 if $x \;\rho\; y$ and is 0 otherwise.

Example:
Consider the relation ρ of the first example, displayed graphically above. There are three ordered pairs in ρ. So there are three entries in M_ρ which are equal to $\mathbf{1}$. The matrix M_ρ is shown below.

$$M_\rho \quad = \quad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

End of example

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side and along the top of the matrix are optional, as they are not part of the matrix. They are only displayed to make it easier to identify the ordered pair in $X \times Y$ associated with each entry of the matrix.

Exercise:
Consider once again the relation between $X = \{1, 2, 3\}$ and $Y = \{1, 3, 5, 9, 11\}$ defined by:

$$x \ \rho \ y \text{ if } 5 - x > y$$

Construct a graphical representation and a logical matrix representation of this relation.

End of exercise

Exercise:
Modify the last exercise by defining a re-

lation $\boldsymbol{\sigma}$ between $\boldsymbol{X}$ and $\boldsymbol{Y}$ as follows:

$$\boldsymbol{x} \ \boldsymbol{\sigma} \ \boldsymbol{y} \text{ if } \mathbf{5} - \boldsymbol{x} < \boldsymbol{y}$$

Construct a graphical representation and a logical matrix representation of this relation.

End of exercise

Some Special Relations

Note that **every** subset of $\boldsymbol{X} \times \boldsymbol{Y}$ is a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$. In particular, the smallest and largest subsets of $\boldsymbol{X} \times \boldsymbol{Y}$, namely $\boldsymbol{\varnothing}$ and $\boldsymbol{X} \times \boldsymbol{Y}$, are examples of relations between $\boldsymbol{X}$ and $\boldsymbol{Y}$. The null set can be called the *empty relation* between $\boldsymbol{X}$ and $\boldsymbol{Y}$, while the Cartesian product $\boldsymbol{X} \times \boldsymbol{Y}$ may be called the *universal relation* between $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Note also that if $\boldsymbol{\rho}$ is a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ then

$$\boldsymbol{\varnothing} \subseteq \boldsymbol{\rho} \subseteq \boldsymbol{X} \times \boldsymbol{Y}.$$

In the logical matrix representation of the empty relation, every entry is $\mathbf{0}$. At the opposite extreme, the universal relation is represented by a logical matrix in which every entry is $\mathbf{1}$.

Another relation of great importance is the *inverse relation* of any given relation. Let $\boldsymbol{\rho}$ be a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$. The inverse relation $\boldsymbol{\rho}^{-1}$ is defined as follows:

$$\boldsymbol{\rho}^{-1} = \{(\boldsymbol{y}, \boldsymbol{x}) \in \boldsymbol{Y} \times \boldsymbol{X} : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\rho}\}$$

That is, $(\boldsymbol{y}, \boldsymbol{x}) \in \boldsymbol{\rho}^{-1}$ if and only if $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\rho}$. So $\boldsymbol{\rho}^{-1}$ has the same pairs as $\boldsymbol{\rho}$, but each ordered pair is written in the reverse order. Clearly, $\boldsymbol{\rho}^{-1}$ is a relation between $\boldsymbol{Y}$ and $\boldsymbol{X}$.

Worked Example:
Consider the relation $\boldsymbol{\rho}$ of the first example:

$$\boldsymbol{\rho = \{(a, 2), (c, 1), (c, 3)\}.}$$

It is a relation between the sets $\boldsymbol{X} = \{\boldsymbol{a, b, c, d}\}$ and $\boldsymbol{Y} = \{\boldsymbol{1, 2, 3}\}$.

The inverse of $\boldsymbol{\rho}$ is the relation $\boldsymbol{\rho}^{-1} = \{(\boldsymbol{2, a}), (\boldsymbol{1, c}), (\boldsymbol{3, c})\} = \{(\boldsymbol{1, c}), (\boldsymbol{2, a}), (\boldsymbol{3, c})\}$.

End of worked example

Worked Example:
What is the inverse $\boldsymbol{\varnothing}^{-1}$ of the empty relation $\boldsymbol{\varnothing}$ between sets $\boldsymbol{X}$ and $\boldsymbol{Y}$?

We get all the elements of $\boldsymbol{\varnothing}^{-1}$ by reversing every ordered pair in $\boldsymbol{\varnothing}$. But there aren't any. So $\boldsymbol{\varnothing}^{-1}$ is empty; that is, $\boldsymbol{\varnothing}^{-1} = \boldsymbol{\varnothing}$.

End of worked example

Exercise:
What is the inverse of the universal relation $\boldsymbol{X} \times \boldsymbol{Y}$ between two sets $\boldsymbol{X}$ and $\boldsymbol{Y}$?

End of exercise

There is a close relationship between the logical matrix representation of a relation and that of its inverse. It depends on the idea of the *transpose* of a matrix.

Let M be a matrix with m rows and n columns. Such a matrix is called an $(m \times n)$-matrix. The *transpose* M^T of M is a matrix that can be constructed from M in the following way. We take the rows of M, one after another, and write them down as columns. This procedure creates the columns of M^T. So the first row of M becomes the first column of M^T, the the second row of M becomes the second column of M^T, and so on. Clearly, the resulting matrix M^T is an $(n \times m)$-matrix.

Then **the logical matrix representing ρ^{-1} is the transpose of the logical matrix representation of ρ**. That is,

$$M_{\rho^{-1}} = M_\rho{}^T.$$

Worked Example:
Consider the relation ρ of the first example:

$$\rho = \{(a, 2), (c, 1), (c, 3)\}.$$

We have seen that its inverse is the relation

$$\rho^{-1} = \{(1, c), (2, a), (3, c)\}.$$

We have also seen that ρ has logical matrix

$$M_\rho = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

So its inverse relation ρ^{-1} has logical matrix

$$M_{\rho^{-1}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

End of worked example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly work with relations on sets.

End of formative assessment

Lecture 10

This lecture provides an introduction to:

- **composition of relations**

You should master the material contained in this lecture before moving on to the next lecture.

The Composite Relation

Suppose that $\boldsymbol{\rho}$ is a relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$, and that $\boldsymbol{\sigma}$ is a relation between $\boldsymbol{Y}$ and $\boldsymbol{Z}$.

Then the *composite relation* $\boldsymbol{\sigma \circ \rho}$ is defined as follows:

$$\boldsymbol{\sigma \circ \rho = \{(x, z) : x\ \rho\ y} \text{ and } \boldsymbol{y\ \sigma\ z} \text{ for some } \boldsymbol{y} \text{ in } \boldsymbol{Y\}}$$

So, loosely speaking, $\boldsymbol{x}$ and $\boldsymbol{z}$ are related to each other (by the composite relation $\boldsymbol{\sigma \circ \rho}$) if there is an element $\boldsymbol{y}$ "in between" them having the property that $\boldsymbol{x}$ is related to $\boldsymbol{y}$ (by $\boldsymbol{\rho}$) and $\boldsymbol{y}$ is related to $\boldsymbol{z}$ (by $\boldsymbol{\sigma}$).

The process of getting $\boldsymbol{\sigma \circ \rho}$ from $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ is called *composition of relations.*

Example:
Let $\boldsymbol{X = \{a, b, c, d, e\}}$, $\boldsymbol{Y = \{0, 1, 2\}}$ and $\boldsymbol{Z = \{w, x, y, z\}}$. Let $\boldsymbol{\rho}$ be the relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined by $\boldsymbol{\rho = \{(a, 0), (c, 0), (d, 1), (d, 2), (e, 2)\}}$. Let $\boldsymbol{\sigma}$ be the relation between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ defined by $\boldsymbol{\sigma = \{(0, w), (0, y), (1, x), (2, x)\}}$.

Then the composite relation $\boldsymbol{\sigma \circ \rho}$ is the following set of ordered pairs:

$$\boldsymbol{\{(a, w), (a, y), (c, w), (c, y), (d, x), (e, x)\}}$$

We can say that $\boldsymbol{a}$ *is related to* $\boldsymbol{w}$ *and to* $\boldsymbol{y}$ *via* $\mathbf{0}$, $\boldsymbol{c}$ *is related to* $\boldsymbol{w}$ *and to* $\boldsymbol{y}$ *via* $\mathbf{0}$, $\boldsymbol{d}$ *is related to* $\boldsymbol{x}$ *via* $\mathbf{1}$ *and via* $\mathbf{2}$, *and* $\boldsymbol{e}$ *is related to* $\boldsymbol{x}$ *via* $\mathbf{2}$.

Figure 1 displays ρ and σ, while Figure 2 is the graphical representation of $\sigma \circ \rho$.

End of example

Example:
Let Z, Y and X represent three successive generations of a family. So each person in Y is the child of someone in Z, and each person in X is the child of someone in Y.

Let $x\ \rho\ y$ mean x *is a child of* y, where $x \in X$ and $y \in Y$. Let $y\ \sigma\ z$ mean y *is a child of* z, where $y \in Y$ and $z \in Z$.

Then $\sigma \circ \rho$ is the set

$$\{(x, z) \in X \times Z : \text{ for some } y \in Y, \; x \text{ is a child of } y \text{ and } y \text{ is a child of } z\}.$$

Equivalently, $\sigma \circ \rho = \{(x, z) \in X \times Z : x$ is a *grandchild* of $z\}$.

End of example

Worked Example:
Let X be the set $\{0, 1, 2, 3\}$ and let $Y = \{3, 6, 9, 12\}$. Let Z be the set consisting of the first ten even positive integers.

Suppose that the relation ρ between X and Y is defined by $x\ \rho\ y$ if $3x = y$, and that the relation σ between Y and Z is defined by $y\ \sigma\ z$ if $2y = z$. Write down the composite relation $\sigma \circ \rho$ as a set of ordered pairs from $X \times Z$. If $(x, z) \in \sigma \circ \rho$,

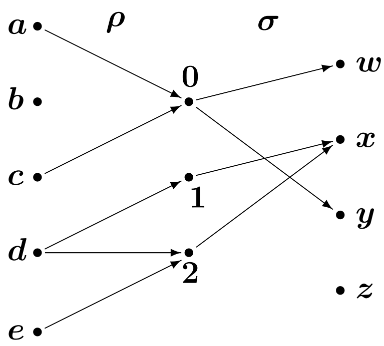

Figure 1: Two Relations.

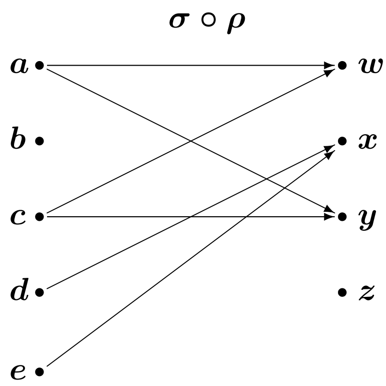

Figure 2: The Composite Relation.

what mathematical relationship exists between $\boldsymbol{x}$ and $\boldsymbol{z}$?

To find the answer, let's suppose that $\boldsymbol{x} \in \boldsymbol{X}$ and $\boldsymbol{z} \in \boldsymbol{Z}$ and that $(\boldsymbol{x}, \boldsymbol{z}) \in \boldsymbol{\sigma} \circ \boldsymbol{\rho}$. Then there exists $\boldsymbol{y} \in \boldsymbol{Y}$ such that $\boldsymbol{x}\ \boldsymbol{\rho}\ \boldsymbol{y}$ and $\boldsymbol{y}\ \boldsymbol{\sigma}\ \boldsymbol{z}$. So $\mathbf{3}\boldsymbol{x} = \boldsymbol{y}$ and $\mathbf{2}\boldsymbol{y} = \boldsymbol{z}$. Therefore $\boldsymbol{z} = \mathbf{2}\boldsymbol{y} = \mathbf{2}(\mathbf{3}\boldsymbol{x}) = \mathbf{6}\boldsymbol{x}$. So $\boldsymbol{z}$ is an even number between 2 and 20 (inclusive) which is six times some integer between 0 and 3.

The only multiples of 6 between 2 and 20 are 6, 12 and 18. Firstly, $\mathbf{6} = \mathbf{6} \cdot \mathbf{1}$ and $\mathbf{1} \in \boldsymbol{X}$. Next, $\mathbf{12} = \mathbf{6} \cdot \mathbf{2}$ and $\mathbf{2} \in \boldsymbol{X}$. Finally, $\mathbf{18} = \mathbf{6} \cdot \mathbf{3}$ and $\mathbf{3} \in \boldsymbol{X}$.

But to be sure that this gives us ordered pairs in $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$, we need to check that there is a $\boldsymbol{y}$ corresponding to each $(\boldsymbol{x}, \boldsymbol{z})$-pair. Can we get from 1 to 6 via some element $\boldsymbol{y}$ of $\boldsymbol{Y}$? Yes; $(\mathbf{1}, \mathbf{3}) \in \boldsymbol{\rho}$ and $(\mathbf{3}, \mathbf{6}) \in \boldsymbol{\sigma}$. So $(\mathbf{1}, \mathbf{6}) \in \boldsymbol{\sigma} \circ \boldsymbol{\rho}$. Similarly, we can get from 2 to 12 via 6 and we can get from 3 to 18 via 9. So $(\mathbf{2}, \mathbf{12})$ and $(\mathbf{3}, \mathbf{18})$ both belong to $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$. There are no other possible ordered pairs in $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$. So

$$\boldsymbol{\sigma} \circ \boldsymbol{\rho} = \{(\mathbf{1}, \mathbf{6}), (\mathbf{2}, \mathbf{12}), (\mathbf{3}, \mathbf{18})\}.$$

As an alternative approach, we could use the graphical representations of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ to construct the composite relation. The approach described above has the advantage that it could be modified for use with

much larger sets.

End of worked example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly determine what pairs are in a composite relation.

End of formative assessment

Lecture 11

This lecture provides a review of matrix multiplication.

You should master the material contained in this lecture before moving on to the next lecture.

Matrix Multiplication

Suppose that $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are two relations for which a composite relation $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ exists. Then from the logical matrix representations of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ we can obtain the logical matrix representation of $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$, even if we don't know $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ explicitly as a set of ordered pairs.

The new matrix can be obtained by *Boolean matrix multiplication.* Before describing this, we firstly give the details of the usual multiplication of matrices.

Recall that a *matrix* $\boldsymbol{M}$ is a rectangular array of numbers or other elements of some set, with the array being enclosed in round or square brackets. The numbers or other elements are the *entries* of the matrix. Entries at the same horizontal level form a *row*, while entries vertically aligned with each other form a *column*. Rows are numbered downwards (so that the top row is the *first row*), while columns are numbered from left to right (so that the leftmost column is the *first column*). A matrix with $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns is called an $\boldsymbol{(m \times n)}$-matrix, and is said to have *order* $\boldsymbol{m \times n}$.

A *row matrix* or *row vector* is a matrix consisting of a single row. A *column matrix* or *column vector* is a matrix consisting of a single column. The *length* of a row or column vector is the number of entries in it.

To describe matrix multiplication we start by multiplying a row vector and a column

vector.

Suppose that a row vector

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}$$

and a column vector

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

have the same length n. Then they can be multiplied in the following way:

$$\begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \dots & \boldsymbol{a_n} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \vdots \\ \boldsymbol{b_n} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a_1\,b_1 + a_2\,b_2 + \cdots + a_n\,b_n} \end{pmatrix} \tag{1}$$

The RHS of Equation 1 is called the *dot product* of the two vectors on the left (although the dot between them is usually omitted). It is a matrix with just a single entry; that is, it is a $(\mathbf{1 \times 1})$-matrix.

Example:

We give an example involving a row vector of length 5 and a column vector of length 5.

Multiplying

$$\begin{pmatrix} \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{1} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{6} \\ \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{10} \end{pmatrix}$$

gives

$$\left(\mathbf{5 \cdot 6 + 4 \cdot 7 + 3 \cdot 8 + 2 \cdot 9 + 1 \cdot 10} \right)$$

which equals

$$\left(\mathbf{30 + 28 + 24 + 18 + 10}\right) = \left(\mathbf{110}\right).$$

End of example

Now the dot product can be extended to a multiplication operation between any two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ which are *compatible for multiplication*. This means that the length of the rows in $\boldsymbol{A}$ equals the length of the columns in $\boldsymbol{B}$.

Equivalently, the number of columns in $\boldsymbol{A}$ equals the number of rows in $\boldsymbol{B}$.

To get $\boldsymbol{AB}$, we begin by constructing its top row. We evaluate the individual entries in this row, going from left to right. We get each entry as the dot product of the first row of $\boldsymbol{A}$ with one of the columns of $\boldsymbol{B}$, starting from the left. Each dot product is written as a number or element, without any brackets around it. These dot products form the top row of $\boldsymbol{AB}$.

To construct the second row of $\boldsymbol{AB}$, we get the dot products of the second row of $\boldsymbol{A}$ with each column of $\boldsymbol{B}$; and so on.

Finally, from the last row of $\boldsymbol{A}$ and all the columns of $\boldsymbol{B}$ we produce all the entries in the last row of $\boldsymbol{AB}$.

So $\boldsymbol{AB}$ has the same number of rows as $\boldsymbol{A}$ (and the same number of columns as $\boldsymbol{B}$).

That is, *the product of an* $(\boldsymbol{m} \times \boldsymbol{n})$*-matrix and an* $(\boldsymbol{n} \times \boldsymbol{k})$*-matrix is an* $(\boldsymbol{m} \times \boldsymbol{k})$*-matrix.*

Example:
Let $\boldsymbol{A}$ be the $(\mathbf{2} \times \mathbf{4})$-matrix

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

and let $\boldsymbol{B}$ be the $(\mathbf{4} \times \mathbf{3})$-matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{-3} & \mathbf{2} \\ \mathbf{-2} & \mathbf{4} & \mathbf{0} \\ \mathbf{5} & \mathbf{0} & \mathbf{2} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \end{pmatrix}.$$

Their product will be the $(\mathbf{2} \times \mathbf{3})$-matrix obtained as follows.

Multiplying the first row

$$\mathbf{1} \quad \mathbf{2} \quad \mathbf{0} \quad \mathbf{-1}$$

of $\boldsymbol{A}$ and the first column

$$\begin{matrix} \mathbf{0} \\ \mathbf{-2} \\ \mathbf{5} \\ \mathbf{3} \end{matrix}$$

of $\boldsymbol{B}$ gives

$$\mathbf{1 \cdot 0 + 2 \cdot (-2) + 0 \cdot 5 + (-1) \cdot 3}$$

which equals

$$\mathbf{0 - 4 + 0 - 3 \;=\; -7}\ .$$

Similarly, multiplying the first row of $\boldsymbol{A}$ with the second and third columns of $\boldsymbol{B}$ gives $\mathbf{3}$ and $\mathbf{3}$ respectively.

Multiplying the second row

$$\mathbf{3 \quad -1 \quad 1 \quad 0}$$

of $\boldsymbol{A}$ and each of the three columns of $\boldsymbol{B}$ gives $\mathbf{7}$, $\mathbf{-13}$ and $\mathbf{8}$ in that order.

So the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{AB} = \begin{pmatrix} \mathbf{-7} & \mathbf{3} & \mathbf{3} \\ \mathbf{7} & \mathbf{-13} & \mathbf{8} \end{pmatrix}.$$

End of example

As mentioned above, multiplying an $(\boldsymbol{m} \times \boldsymbol{n})$-matrix $\boldsymbol{A}$ and an $(\boldsymbol{n} \times \boldsymbol{k})$-matrix $\boldsymbol{B}$ gives an $(\boldsymbol{m} \times \boldsymbol{k})$-matrix $\boldsymbol{AB}$. What happens if we multiply them in the reverse order?

The answer is that the product matrix $\boldsymbol{BA}$ might not even exist. Specifically, $\boldsymbol{BA}$ exists if and only if $\boldsymbol{k} = \boldsymbol{m}$. When $\boldsymbol{k} = \boldsymbol{m}$ we are multiplying an $(\boldsymbol{n} \times \boldsymbol{m})$-matrix $\boldsymbol{B}$ and an $(\boldsymbol{m} \times \boldsymbol{n})$-matrix $\boldsymbol{A}$, and the product $\boldsymbol{BA}$ is an $(\boldsymbol{n} \times \boldsymbol{n})$-matrix.

Example:
In the last example, $\boldsymbol{A}$ had order $\mathbf{2} \times \mathbf{4}$ while $\boldsymbol{B}$ had order $\mathbf{4} \times \mathbf{3}$. So the product $\boldsymbol{AB}$ had order $\mathbf{2} \times \mathbf{3}$.

But if we tried to obtain $\boldsymbol{BA}$, we would be multiplying a $(\mathbf{4} \times \mathbf{3})$-matrix $\boldsymbol{B}$ and a $(\mathbf{2} \times \mathbf{4})$-matrix $\boldsymbol{A}$. Because the number of columns in $\boldsymbol{B}$ (which is $\mathbf{3}$) doesn't equal the number of rows in $\boldsymbol{A}$ (which is $\mathbf{2}$), the product $\boldsymbol{BA}$ doesn't exist.

End of example

From the last example, we see that it can happen that $\boldsymbol{AB}$ exists while $\boldsymbol{BA}$ doesn't.

Even if they both exist, they could be of different orders. And even if they both exist and have the same order, they need not be equal.

The next two examples illustrate these features of matrix multiplication.

Example:
Let $\boldsymbol{A}$ be the $(\mathbf{1} \times \mathbf{2})$-matrix

$$\begin{pmatrix} \mathbf{5} & \mathbf{-2} \end{pmatrix}$$

and $\boldsymbol{B}$ be the $(\mathbf{2} \times \mathbf{1})$-matrix

$$\begin{pmatrix} \mathbf{3} \\ \mathbf{4} \end{pmatrix}.$$

Then $\boldsymbol{AB}$ is the $(\mathbf{1} \times \mathbf{1})$-matrix

$$\begin{pmatrix} \mathbf{5} \cdot \mathbf{3} + (\mathbf{-2}) \cdot \mathbf{4} \end{pmatrix}$$

which equals

$$\begin{pmatrix} \mathbf{7} \end{pmatrix}.$$

In contrast, $\boldsymbol{BA}$ is the $(\mathbf{2} \times \mathbf{2})$-matrix

$$\begin{pmatrix} \mathbf{3 \cdot 5} & \mathbf{3 \cdot (-2)} \\ \mathbf{4 \cdot 5} & \mathbf{4 \cdot (-2)} \end{pmatrix}$$

which equals

$$\begin{pmatrix} \mathbf{15} & \mathbf{-6} \\ \mathbf{20} & \mathbf{-8} \end{pmatrix}.$$

End of example

We have seen that $\boldsymbol{AB}$ and $\boldsymbol{BA}$ both exist if and only if $\boldsymbol{A}$ is an $(\boldsymbol{m} \times \boldsymbol{n})$-matrix while $\boldsymbol{B}$ is an $(\boldsymbol{n} \times \boldsymbol{m})$-matrix. When this situation arises, the product $\boldsymbol{AB}$ will be an $(\boldsymbol{m} \times \boldsymbol{m})$-matrix while the reverse product $\boldsymbol{BA}$ will be an $(\boldsymbol{n} \times \boldsymbol{n})$-matrix.

The only way that these two product matrices can have the same order is for $\boldsymbol{m}$ and $\boldsymbol{n}$ to be equal. Then all four matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are $(\boldsymbol{n} \times \boldsymbol{n})$-matrices. Such matrices are called *square* matrices.

The product of any two square matrices of the same order is another square matrix of the same order. So for every positive integer $\boldsymbol{n}$ we say that the collection of all $(\boldsymbol{n} \times \boldsymbol{n})$-matrices is a *closed system* under matrix multiplication.

Worked Example:
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be the $(\mathbf{2} \times \mathbf{2})$-matrices

$$\begin{pmatrix} \mathbf{-2} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{5} & \mathbf{-4} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$$

respectively. What is $\boldsymbol{AB}$? What is $\boldsymbol{BA}$?

We get $\boldsymbol{AB}$ to be

$$\begin{pmatrix} \mathbf{-10} & \mathbf{8} \\ \mathbf{14} & \mathbf{-12} \end{pmatrix}$$

while $\boldsymbol{BA}$ is

$$\begin{pmatrix} \mathbf{-22} & \mathbf{-4} \\ \mathbf{2} & \mathbf{0} \end{pmatrix}.$$

End of worked example

From the last Worked Example and the two Examples before it, we see that it is **not** always the case that $\boldsymbol{AB} = \boldsymbol{BA}$ for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Often we say that the operation of matrix multiplication is **not** *commutative*. In fact, matrix multiplication is only a *partial* operation on the set of all matrices with real entries, because the products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ do not always exist.

Because $\boldsymbol{AB}$ and $\boldsymbol{BA}$ often have different meanings, it is ambiguous to speak of "multiplying $\boldsymbol{A}$ by $\boldsymbol{B}$". Instead, we use the more precise expressions *premultiplication*

and *postmultiplication.* We say that $\boldsymbol{AB}$ is obtained through *postmultiplying* $\boldsymbol{A}$ by $\boldsymbol{B}$, or (equivalently) through *premultiplying* $\boldsymbol{B}$ by $\boldsymbol{A}$.

In the next lecture we discuss Boolean matrix multiplication and how it is linked to the composition of relations.

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly determine whether or not two matrices can be multiplied, and to find their product where possible .

End of formative assessment

Lecture 12

This lecture provides an introduction to:

- **Boolean matrix multiplication**
- **the logical matrix representation of a composite relation**

You should master the material contained in this lecture before moving on to the next lecture.

Boolean Matrix Products

We have seen that two matrices can be multiplied together, provided that they are compatible for multiplication. The matrix product is obtained by a procedure which involves both multiplication and addition among the entries of the two matrices. Each entry in the product matrix is the sum of certain products of entries of the left-hand matrix with entries of the right-hand matrix.

A matrix in which every entry is 0 or 1 is sometimes called a *Boolean* matrix. Now we give an example of the multiplication of two Boolean matrices.

Example:
Let $\boldsymbol{A}$ be the $\mathbf{(2 \times 3)}$-matrix which is shown below.

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

Let $\boldsymbol{B}$ be the $\mathbf{(3 \times 2)}$-matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

Their product will be the $\mathbf{(2 \times 2)}$-matrix obtained as follows.

Multiplying the first row

$$\mathbf{1 \quad 0 \quad 0}$$

of $\boldsymbol{A}$ and the first column

$$\begin{matrix}\mathbf{0}\\\mathbf{1}\\\mathbf{1}\end{matrix}$$

of $\boldsymbol{B}$ gives

$$\mathbf{1\cdot 0 + 0 \cdot 1 + 0 \cdot 1}$$

which equals

$$\mathbf{0+0+0 \;=\; 0\ .} \tag{1}$$

Multiplying the first row

$$\begin{matrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\end{matrix}$$

of $\boldsymbol{A}$ and the second column

$$\begin{matrix}\mathbf{1}\\\mathbf{1}\\\mathbf{1}\end{matrix}$$

of $\boldsymbol{B}$ gives

$$\mathbf{1\cdot 1 + 0 \cdot 1 + 0 \cdot 1}$$

which equals

$$\mathbf{1+0+0 \;=\; 1\ .} \tag{2}$$

Multiplying the second row

$$\begin{matrix}\mathbf{1} & \mathbf{1} & \mathbf{1}\end{matrix}$$

of $\boldsymbol{A}$ and the first column

$$\begin{matrix}\mathbf{0}\\\mathbf{1}\\\mathbf{1}\end{matrix}$$

of $\boldsymbol{B}$ gives

$$\mathbf{1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1}$$

which equals

$$\mathbf{0 + 1 + 1 \ = \ 2 \ .} \tag{3}$$

Multiplying the second row

$$\mathbf{1 \quad 1 \quad 1}$$

of $\boldsymbol{A}$ and the second column

$$\begin{matrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{matrix}$$

of $\boldsymbol{B}$ gives

$$\mathbf{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1}$$

which equals

$$\mathbf{1 + 1 + 1 \ = \ 3 \ .} \tag{4}$$

So the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{AB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{pmatrix} .$$

End of example

Each of (1), (2), (3) and (4) in the last example describe additions. Each number being added is either 0 or 1.

In such a situation *we can replace ordinary addition by* **Boolean** *addition.* The multiplications which took place just prior to the additions can be regarded as Boolean multiplications. The resulting matrix is the *Boolean matrix product* of the two Boolean matrices.

Example:
We revisit the last example. Replacing ordinary addition by Boolean addition does not change the sums in (1) and (2) above, but it does change (3) and (4). Instead of (3) we now have

$$\mathbf{0+1+1 \;=\; 1} \tag{5}$$

and instead of (4) we have

$$\mathbf{1+1+1 \;=\; 1\ .} \tag{6}$$

The sums in (1), (2), (5) and (6) above give us the *Boolean* product of $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\boldsymbol{AB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

End of example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly determine the Boolean product of two Boolean matrices.

End of formative assessment

The Matrix of Composition

Now we are able to state the main result of this lecture. The proof is a little tedious, and is omitted.

Theorem:
Let $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ be two relations with logical matrix representations $\boldsymbol{M_\rho}$ and $\boldsymbol{M_\sigma}$ respectively. Then the logical matrix representation of the composite relation $\boldsymbol{\sigma \circ \rho}$ is the Boolean matrix product of the logical matrices representing the two relations $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$:

$$\boldsymbol{M_{\sigma \circ \rho} = M_\rho \cdot M_\sigma}$$

where the product on the right is the **Boolean** product of the two matrices.

End of theorem

Example:
Let $\boldsymbol{X = \{a, b, c, d, e\}}$, $\boldsymbol{Y = \{0, 1, 2\}}$

and $\boldsymbol{Z} = \{\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$. Let $\boldsymbol{\rho}$ be the relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined by $\boldsymbol{\rho} = \{(\boldsymbol{a}, \mathbf{0}), (\boldsymbol{c}, \mathbf{0}), (\boldsymbol{d}, \mathbf{1}), (\boldsymbol{d}, \mathbf{2}), (\boldsymbol{e}, \mathbf{2})\}$. Let $\boldsymbol{\sigma}$ be the relation between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ defined by $\boldsymbol{\sigma} = \{(\mathbf{0}, \boldsymbol{w}), (\mathbf{0}, \boldsymbol{y}), (\mathbf{1}, \boldsymbol{x}), (\mathbf{2}, \boldsymbol{x})\}$.

Then the composite relation $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ is the following set of ordered pairs:

$$\{(\boldsymbol{a}, \boldsymbol{w}), (\boldsymbol{a}, \boldsymbol{y}), (\boldsymbol{c}, \boldsymbol{w}), (\boldsymbol{c}, \boldsymbol{y}), (\boldsymbol{d}, \boldsymbol{x}), (\boldsymbol{e}, \boldsymbol{x})\}$$

The logical matrix representations of $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ are

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

and

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

respectively. It is routine to verify that the Boolean product of the first and second matrices is the third matrix.

End of example

Worked Example:
Let $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$, let $\boldsymbol{Y}$ be a three-element set and let $\boldsymbol{Z} = \{\boldsymbol{x}, \boldsymbol{y}\}$. Let $\boldsymbol{\rho}$ be the relation between $\boldsymbol{X}$ and $\boldsymbol{Y}$ having logical matrix representation

$$\boldsymbol{M_\rho} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Let $\boldsymbol{\sigma}$ be the relation between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ having logical matrix representation

$$\boldsymbol{M_\sigma} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Find the Boolean product of the two logical matrices, and use it to find the composite relation $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ as a set of ordered pairs.

Multiplying the two matrices (using Boolean multiplication of the appropriate entries and Boolean addition of the resulting products) gives

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

This is the logical matrix for the relation $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ between the sets $\boldsymbol{X} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ and $\boldsymbol{Z} = \{\boldsymbol{x}, \boldsymbol{y}\}$. So $\boldsymbol{\sigma} \circ \boldsymbol{\rho}$ is the set

$$\{(\boldsymbol{a}, \boldsymbol{x}), (\boldsymbol{a}, \boldsymbol{y}), (\boldsymbol{b}, \boldsymbol{x}), (\boldsymbol{b}, \boldsymbol{y}), (\boldsymbol{c}, \boldsymbol{y}), (\boldsymbol{d}, \boldsymbol{x}), (\boldsymbol{d}, \boldsymbol{y})\}.$$

End of worked example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly use the Boolean product of two logical matrices to construct a composite relation.

End of formative assessment

Lecture 13

This lecture provides an introduction to:

- relations on a set
- representations of relations on a set
- partitions and equivalence classes

You should master the material contained in this lecture before moving on to the next lecture.

Relations on a Set

We are familiar with the idea of a *relation ρ between two sets X and Y*. But it can happen that X and Y are one and the same. In such cases, ρ is a relation between X and X. We say that *ρ is a relation on the set X*.

So a relation on X is a subset of the Cartesian product of X with itself, which is the set X^2 of all ordered pairs (x, y) where both x and y are elements of X.

All the results about relations between two sets apply to relations on a set. For example, we have seen that the number of relations between a set X and a set Y is:

$$2^{|X|\cdot|Y|}$$

So, in particular, the number of relations on a set X is:

$$2^{|X|\cdot|X|} = 2^{|X|^2}$$

Also, we know that a relation ρ between two sets X and Y has a logical matrix representation M_ρ, where the number of rows in M_ρ is $|X|$ and the number of columns is $|Y|$. So if $X = Y$ then M_ρ has $|X|$ rows and $|X|$ columns, which means that M_ρ is a **square** matrix.

Example:

Let $\boldsymbol{X} = \{1, 2, 3\}$. Since $\boldsymbol{X}$ is a 3-element set, the number of relations on $\boldsymbol{X}$ is

$$2^{3\cdot 3} = 2^9 = 512.$$

Here is an example of one of those relations. Let $\boldsymbol{\rho}$ be the set

$$\{(2, 1), (3, 1), (3, 2), (3, 3)\}.$$

Then $\boldsymbol{\rho}$ is a relation on $\boldsymbol{X}$.

Here is the logical matrix representation $M_{\boldsymbol{\rho}}$ of $\boldsymbol{\rho}$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

A graphical representation of $\boldsymbol{\rho}$ is shown in Figure 1.

Notice that the points in the graphical representation are duplicated; that is, each point representing an element of $\boldsymbol{X}$ appears twice.

This duplication can be avoided. We can just draw one point for each element of $\boldsymbol{X}$, and put an arrow from a point $\boldsymbol{x}$ of $\boldsymbol{X}$ to another point $\boldsymbol{y}$ of $\boldsymbol{X}$ if $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\rho}$. The directed graph that we get is shown in Figure 2.

End of example

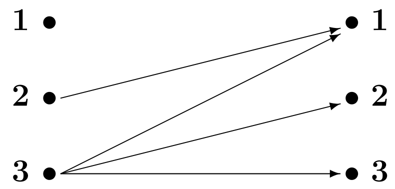

Figure 1: A Relation between a Set and itself.

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly find the relation on a set represented by a square Boolean matrix.

End of formative assessment

Some Special Relations

The *identity relation* ι_X on a set X consists of all pairs of the form (x, x). If $|X| = n$ then the logical matrix repre-

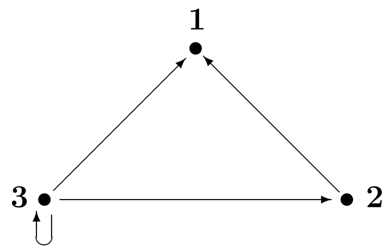

Figure 2: A Relation on a Set.

sentation of ι_X is the $(n \times n)$-identity matrix I_n.

The identity relation can be represented graphically by drawing a point for each element of X and putting a looping arrow from each point to itself.

Example:
Let $X = \{a, b, c\}$. Then ι_X is the set

$$\{(a, a), (b, b), (c, c)\}.$$

Its logical matrix representation is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A graphical representation is shown in Figure 3.

End of example

The *universal relation* ω_X on a set X consists of **all** ordered pairs from X^2 of the form (x, y). That is, $\omega_X = X^2$.

In the logical matrix representation of ω_X, the matrix is square (having $|X|$ rows and $|X|$ columns) and every entry is 1. In its graphical representation, there are arrows from every element of X to every other element of X (and from each element of X to itself). So the directed graph often looks rather cluttered.

Example:
Let X be the set $\{a, b, c\}$, as in the last example. Then ω_X is the set

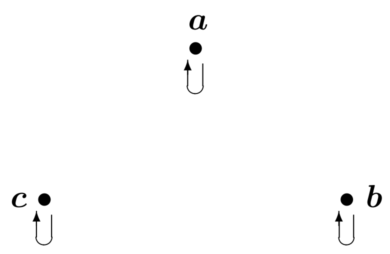

Figure 3: The Identity Relation on a Set.

$$\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}.$$

Its logical matrix representation is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

A graphical representation is shown in Figure 4.

End of example

Note that the *empty relation* $\varnothing$ can be defined on any set X, in which case it is sometimes denoted by λ_X. In the matrix representation, every entry is 0. The graphical representation exhibits a set of points (one for each element of X) and no arrows at all.

Then, for any relation ρ on X,

$$\lambda_X \subseteq \rho \subseteq \omega_X.$$

This is a special case of a more general statement made in Lecture 9.

So all relations on X contain λ_X and are contained in ω_X. As we shall see, those relations on X which contain ι_X are also of special interest.

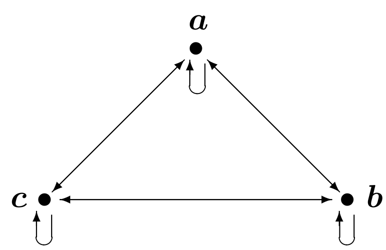

Figure 4: The Universal Relation on a Set.

Properties of Relations

Certain kinds of relations on a set are of particular importance. Relations that are are *reflexive*, *symmetric* and *transitive* are called *equivalence relations*, and those that are reflexive, *antisymmetric* and transitive are called *partial order relations*. We define these properties one by one.

A relation ρ on a set X is said to be *reflexive* if every pair of the form (x, x) belongs to ρ (where $x \in X$). That is,

$$x \in X \Rightarrow x \ \rho \ x.$$

Equivalently, ρ is reflexive if and only if $\iota_X \subseteq \rho$.

A relation ρ on a set X is said to be *symmetric* if whenever an ordered pair (x, y) is in ρ then so is the reverse ordered pair (y, x). That is,

$$x \ \rho \ y \Rightarrow y \ \rho \ x.$$

Equivalently, ρ is symmetric if and only if $p^{-1} = \rho$. (Recall that the inverse of a relation was defined in Lecture 9.)

A relation ρ on a set X is said to be *transitive* if whenever two ordered pairs (x, y) and (y, z) are both in ρ then so is the ordered pair (x, z). That is,

$$\left.\begin{array}{l} x \ \rho \ y \\ y \ \rho \ z \end{array}\right\} \Rightarrow x \ \rho \ z.$$

Equivalently, ρ is transitive if and only if

$$\rho \circ \rho \subseteq \rho.$$

A relation ρ on a set X is an *equivalence (relation)* if it is reflexive, symmetric and transitive.

That is, ρ is an equivalence on X if it has the following three properties:

(i) $x \; \rho \; x$ for all $x \in X$ [reflexivity]

(ii) $x \; \rho \; y \Rightarrow y \; \rho \; x$ [symmetry]

(iii) $(x \; \rho \; y$ and $y \; \rho \; z) \Rightarrow x \; \rho \; z$ [transitivity]

If a relation ρ on a set X is an equivalence relation, and if $(x, y) \in \rho$, then as well as saying that x is ρ-related to y we also say that x is ρ-*equivalent* to y. Because of symmetry, we can also say that x and y are ρ-equivalent to each other. Sometimes we say more simply that x *and* y *are equivalent.*

Worked Example:
Let $X = \{a, b, c, d\}$ and let ρ be the relation

$$\{(a, a), (a, b), (b, c), (c, c), (d, d)\}$$

on X. Which of the properties of reflexivity, symmetry and transitivity hold for this relation? Is it an equivalence relation?

We first check for reflexivity. Since the set X has four elements a, b, c and d, a re-

lation on this set can only be reflexive if it includes the four pairs (a, a), (b, b), (c, c) and (d, d).

Looking at ρ we see that the pair (b, b) is missing, so that ρ is not reflexive.

Now we look for symmetry. We need to look at each ordered pair (x, y) in ρ and make sure that the reverse ordered pair (y, x) is also in ρ. There's no need to check pairs of the form (x, x), since the reverse pair is also (x, x). So we only need to check ordered pairs of the form (x, y) where $x \neq y$.

Examining ρ, we find the ordered pairs of this form are (a, b) and (b, c). To be symmetric, ρ must therefore also have (b, a) and (c, b) in it. But they are not there. So there are two reasons for ρ not being symmetric.

Finally we check for transitivity. We need to find two ordered pairs in ρ having a particular connection between them, in that the second coordinate of one of them has to be the first coordinate of the other one. That is, we want to find two ordered pairs (x, y) and (y, z). For transitivity, every time we find two such ordered pairs then we must also be able to find the third ordered pair (x, z).

There is no point in looking at groups of ordered pairs of the form (x, x) and (x, y) or the form (x, y) and (y, y), since in both cases the "third" pair is (x, y) which

is in $\boldsymbol{\rho}$. So for transitivity to fail it must be possible to find in $\boldsymbol{\rho}$ two ordered pairs of the form $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{y}, \boldsymbol{z})$ where $\boldsymbol{x} \neq \boldsymbol{y}$ and $\boldsymbol{y} \neq \boldsymbol{z}$ and such that the ordered pair $(\boldsymbol{x}, \boldsymbol{z})$ is not in $\boldsymbol{\rho}$.

Such a situation does arise in this example. Looking in $\boldsymbol{\rho}$ we find $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{b}, \boldsymbol{c})$, which means that for transitivity $\boldsymbol{\rho}$ must also have $(\boldsymbol{a}, \boldsymbol{c})$ in it. But $(\boldsymbol{a}, \boldsymbol{c})$ is not there. So $\boldsymbol{\rho}$ is not transitive.

Since all three properties fail, $\boldsymbol{\rho}$ is not an equivalence relation. (Actually, it only requires one of the properties to fail for $\boldsymbol{\rho}$ to not be an equivalence relation. So after finding that $\boldsymbol{\rho}$ was not reflexive we could have said immediately that $\boldsymbol{\rho}$ is not an equivalence.)

End of worked example

Note that transitivity also fails if a relation has ordered pairs $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{y}, \boldsymbol{x})$ in it but doesn't have $(\boldsymbol{x}, \boldsymbol{x})$ in it. This is illustrated in the next example.

Example:
Let $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and let $\boldsymbol{\sigma}$ be the set

$$\{(\mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{2}, \mathbf{1})\}$$

of ordered pairs.

Here, $\boldsymbol{\sigma}$ is not transitive. Because $(\mathbf{1}, \mathbf{2})$ and $(\mathbf{2}, \mathbf{1})$ are in $\boldsymbol{\sigma}$, so should $(\mathbf{1}, \mathbf{1})$ be; and it is. But also, because $(\mathbf{2}, \mathbf{1})$ and

$(1,2)$ are in σ, $(2,2)$ should be. And it isn't.

The absence of $(2,2)$ also makes σ nonreflexive. However, σ is symmetric.

Since σ is neither reflexive nor transitive, it is not an equivalence relation.

End of example

Worked Example:
Let $X = \{a,b,c,d\}$ and let ρ be the set

$$\{(a,a),(a,b),(a,c),(b,a),(b,b),\\ (b,c),(c,a),(c,b),(c,c),(d,d)\}$$

of ordered pairs. Which of the properties of reflexivity, symmetry and transitivity hold for this relation? Is it an equivalence relation?

Since the set X has just four elements a, b, c and d, and the four pairs (a,a), (b,b), (c,c) and (d,d) all belong to ρ, ρ is reflexive.

Checking each ordered pair in ρ of the form (x,y) where $x \neq y$, we find that its reverse ordered pair (y,x) is also in ρ. For example, (a,b) is in ρ and so is (b,a). Because all of the required reverse ordered pairs are present, ρ is symmetric.

Because ρ is reflexive, to check for transitivity we only need to look for groups of ordered pairs of the form (x,y) and (y,z) where x, y and z are all distinct. Since

(a, b) and (b, c) are in ρ, transitivity requires that (a, c) be also in ρ; and it is. Similarly, the presence of (a, c) and (c, b) requires (a, b), which is also in ρ. Next, because (b, a) and (a, c) are in ρ we need (b, c) to be there too; and it is. We are half-way through checking for transitivity.

In ρ are also (b, c) and (c, a), requiring (b, a) which is also in ρ. The ordered pairs (c, a) and (a, b) require (c, b) to be present; and it is. Finally, from the membership in ρ of (c, b) and (b, a) we need to have (c, a) as an element of ρ; and it is there. So ρ is transitive.

Since ρ is reflexive, symmetric and transitive, it is an equivalence relation.

End of worked example

Example:
The universal relation ω_X on any set X is an equivalence relation.

This is because the three criteria for a relation ρ to be an equivalence all require that certain ordered pairs from X^2 be in ρ. But since every ordered pair from X^2 is in ω_X, all three properties are satisfied. Hence ω_X is an equivalence relation.

End of example

Example:

The identity relation ι_X on a set X is an equivalence relation.

By definition, ι_X consists of all pairs of the form (x, x) where $x \in X$. But any relation having all of those pairs in it is reflexive. Therefore ι_X is reflexive.

Since ι_X includes no pairs of the form (x, y) where $x \neq y$, we are never in the situation of having to look for the reverse ordered pair (y, x). So ι_X is symmetric.

Similarly, since ι_X includes no pairs of the form (x, y) where $x \neq y$, we are never in the situation of having (x, y) and (y, z) both belonging to the relation where $x \neq y$ and $y \neq z$. So there is never a third ordered pair (x, z) being sought. Hence ι_X is transitive.

End of example

Example:
The empty relation λ_X on any nonempty set X is symmetric and transitive, but not reflexive.

Since X is nonempty, there is at least one element x in X. The pair (x, x) is in any reflexive relation, but it is not in λ_X (because there are **no** ordered pairs in λ_X). So λ_X is not reflexive.

But λ_X is symmetric and transitive, for the same reasons that ι_X is reflexive and transitive (as shown in the last example).

End of example

Note that the empty relation on the empty set $\varnothing$ is technically an equivalence relation, though in a completely vacuous way.

Exercise:
Let $X = \{a, b, c\}$. Construct relations on X that are: reflexive and symmetric but not transitive; reflexive and transitive but not symmetric; and symmetric and transitive but not reflexive.

End of exercise

Exercise:
Let $X = \{a, b, c\}$. Construct relations on X that are: reflexive but neither symmetric nor transitive; symmetric but neither reflexive nor transitive; and transitive but neither reflexive nor symmetric.

End of exercise

Notice that often a relation fails to satisfy one of the three properties because certain ordered pairs are missing. This suggests that if we add those missing ordered pairs we can "fix up" the problem, so that the relation will then be reflexive, symmetric or transitive. But in fact we will no longer have the same relation. What we

will have is a relation which is closely associated with the original relation.

The *reflexive closure* ρ^r of a relation ρ is the smallest reflexive relation containing ρ. It is created in the following way. If there are any pairs of the form (x, x) missing from ρ, we simply **enlarge** ρ by adding those pairs. If ρ already has all such pairs, then ρ is reflexive and we leave it as it is.

This means that

$$\rho^r = \rho \cup \iota_X.$$

The *symmetric closure* ρ^s of a relation ρ is the smallest symmetric relation containing ρ. It is created in the following way. If ρ has any ordered pairs of the form (x, y) for which the reverse ordered pair (y, x) is missing, we enlarge ρ by adding those reverse ordered pairs. Otherwise, ρ is symmetric and we leave it unchanged.

This means that

$$\rho^s = \rho \cup \rho^{-1}.$$

The *transitive closure* ρ^t of a relation ρ is the smallest transitive relation containing ρ. We construct it as follows. We look for any group of ordered pairs (x, y) and (y, z) in ρ for which (x, z) is not in ρ, and in every such case we add (x, z) to ρ. If the enlarged relation is transitive, we stop. Otherwise, we repeat the process until eventually we get a transitive relation.

Technically, this means that

$$\rho^t = \rho \cup (\rho \circ \rho) \cup (\rho \circ \rho \circ \rho) \cup \ldots$$

which is just a more compact way of describing the procedure outlined above.

Example:
Let $X = \{a, b, c, d\}$ and let ρ be the relation

$$\{(a, b), (b, c), (c, d)\}.$$

We construct the transitive closure ρ^t of ρ.

A transitive relation that included (a, b) and (b, c) would also include (a, c). So we have to adjoin (a, c) to ρ. Also, because of the presence of (b, c) and (c, d) we need to include (b, d). This give us the relation:

$$\{(a, b), (a, c), (b, c), (b, d), (c, d)\}$$

This is the union of ρ and $\rho \circ \rho$.

But this is still not transitive, because adjoining (a, c) and (b, d) has created the need for at least one new ordered pair. Specifically, we now need (a, d) (because the relation has (a, b) and (b, d) in it, and also because the relation has (a, c) and (c, d) in it). Adjoining (a, d) gives us this relation:

$$\{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$$

This is the union of ρ, $\rho \circ \rho$ and $\rho \circ \rho \circ \rho$.

A careful check confirms that we now have

a transitive relation. So this must be the transitive closure of ρ. That is, ρ^t is the set

$$\{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}$$

of ordered pairs.

End of example

Now let ρ be any relation on a set X. Then we use the notation ρ^* to represent the smallest equivalence relation on X containing ρ. This is called the equivalence relation *generated by* ρ. It can be shown that

$$\rho^* = ((\rho^r)^s)^t.$$

Example:
Let $X = \{a,b,c,d\}$ and let ρ be the set

$$\{(a,a),(a,b),(b,c),(c,c),(d,d)\}$$

of ordered pairs. Find ρ^r, ρ^s, ρ^t and ρ^*.

We have previous seen (in the first worked example) that ρ does not have any of the three properties of reflexivity, symmetry and transitivity.

Since (b,b) is missing, and it is the only pair of the form (x,x) which is missing, we create ρ^r by adjoining (b,b) to ρ. So ρ^r is the set

$$\{(a,a),(a,b),(b,b),(b,c),(c,c),(d,d)\}$$

of ordered pairs.

To get a symmetric relation from $\boldsymbol{\rho}$, we note firstly that it is the absence of $(\boldsymbol{b}, \boldsymbol{a})$ and $(\boldsymbol{c}, \boldsymbol{b})$ which prevents $\boldsymbol{\rho}$ from being symmetric. So we adjoin these to $\boldsymbol{\rho}$. Then $\boldsymbol{\rho}^{\boldsymbol{s}}$ is the set

$$\{(\boldsymbol{a}, \boldsymbol{a}), (\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{b}, \boldsymbol{a}), (\boldsymbol{b}, \boldsymbol{c}), (\boldsymbol{c}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{c}), (\boldsymbol{d}, \boldsymbol{d})\}$$

of ordered pairs.

To create the smallest transitive relation containing $\boldsymbol{\rho}$, we observe that a transitive relation having $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{b}, \boldsymbol{c})$ in it (as $\boldsymbol{\rho}$ does) would also have $(\boldsymbol{a}, \boldsymbol{c})$ in it (which $\boldsymbol{\rho}$ doesn't). So we adjoin $(\boldsymbol{a}, \boldsymbol{c})$ to $\boldsymbol{\rho}$, thereby creating a transitive relation which must be $\boldsymbol{\rho}^{\boldsymbol{t}}$. It is the set

$$\{(\boldsymbol{a}, \boldsymbol{a}), (\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{a}, \boldsymbol{c}), (\boldsymbol{b}, \boldsymbol{c}), (\boldsymbol{c}, \boldsymbol{c}), (\boldsymbol{d}, \boldsymbol{d})\}$$

of ordered pairs.

Now none of the three relations just constructed is an equivalence relation. But we can begin from $\boldsymbol{\rho}^{\boldsymbol{r}}$ and proceed to construct $\boldsymbol{\rho}^{*}$. The lack of symmetry in $\boldsymbol{\rho}^{\boldsymbol{r}}$ is due to the absence of $(\boldsymbol{b}, \boldsymbol{a})$ and $(\boldsymbol{c}, \boldsymbol{b})$, so we adjoin them to $\boldsymbol{\rho}^{\boldsymbol{r}}$ to create $(\boldsymbol{\rho}^{\boldsymbol{r}})^{\boldsymbol{s}}$:

$$\{(\boldsymbol{a}, \boldsymbol{a}), (\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{b}, \boldsymbol{a}), (\boldsymbol{b}, \boldsymbol{b}), (\boldsymbol{b}, \boldsymbol{c}), (\boldsymbol{c}, \boldsymbol{b}), (\boldsymbol{c}, \boldsymbol{c}), (\boldsymbol{d}, \boldsymbol{d})\}$$

Now this relation is reflexive and symmetric, but fails to be transitive because of the absence of $(\boldsymbol{a}, \boldsymbol{c})$ and $(\boldsymbol{c}, \boldsymbol{a})$. Adjoining them gives us the following relation, which

is the composition of $(\rho^r)^s$ with itself:

$$\{(a,a),(a,b),(a,c),(b,a),(b,b),\\ (b,c),(c,a),(c,b),(c,c),(d,d)\}$$

Carefully examining this relation, we find it to be transitive. (This means that it must be the transitive closure $((\rho^r)^s)^t$ of $(\rho^r)^s$.) So finally we have a relation which contains ρ and which is reflexive, symmetric and transitive. Furthermore, it is the smallest such relation. So it is the equivalence relation ρ^* generated by ρ. That is, ρ^* is the set

$$\{(a,a),(a,b),(a,c),(b,a),(b,b),\\ (b,c),(c,a),(c,b),(c,c),(d,d)\}.$$

End of example

Matrix Representations

The properties of reflexivity, symmetry and transitivity can be observed by examining the logical matrix representations of relations having these properties. We have the following results, which we state without proof.

Firstly, if M and N are two matrices of the same order then we say that $M \leq N$ if every entry of M is less than or equal to the corresponding entry of N. We also say that $M < N$ if $M \leq N$ and M is

not equal to $\boldsymbol{N}$.

A relation $\boldsymbol{\rho}$ on a set $\boldsymbol{X}$ of cardinality $\boldsymbol{n}$ is reflexive if and only if every entry in the leading diagonal of $\boldsymbol{M_\rho}$ is 1; equivalently,

$$\boldsymbol{I_n \leq M_\rho}.$$

A relation $\boldsymbol{\rho}$ is symmetric if and only if its logical matrix representation is a symmetric matrix; equivalently,

$$\boldsymbol{M_\rho = {M_\rho}^T}.$$

A relation $\boldsymbol{\rho}$ is transitive if and only if

$$\boldsymbol{M_\rho \cdot M_\rho \leq M_\rho}$$

where the product on the left-hand side is the Boolean product of the two matrices.

Example:
Suppose that the matrix

$$\boldsymbol{M} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

represents a relation $\boldsymbol{\rho}$ on a set $\boldsymbol{X}$. We want to know if $\boldsymbol{\rho}$ is reflexive, if it is symmetric, and if it is transitive.

The leading-diagonal entries are 0, 1 and 0 respectively. Since not all of them are equal to 1, the matrix represents a relation which is not reflexive.

The 1 in the (1,2)-position (where the first row meets the second column) is not

matched by a 1 in the (2,1)-position. This is enough to show that the matrix is not symmetric. Therefore the relation represented by the matrix also fails to be symmetric.

To check for transitivity we need to multiply the matrix by itself (using Boolean multiplication) and compare the result with the original matrix. The Boolean matrix product $\boldsymbol{M} \cdot \boldsymbol{M}$ is

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

which is the same as $\boldsymbol{M}$ except in the (1,3)-position. In $\boldsymbol{M} \cdot \boldsymbol{M}$ there is a 0 in this position, whereas $\boldsymbol{M}$ has a 1 there. So $\boldsymbol{M} \cdot \boldsymbol{M} < \boldsymbol{M}$, which satisfies the criterion for transitivity (that $\boldsymbol{M} \cdot \boldsymbol{M} \leq \boldsymbol{M}$). So the relation $\boldsymbol{\rho}$ is transitive.

End of example

Worked Example:
Suppose that the matrix

$$\boldsymbol{M} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

represents a relation $\boldsymbol{\rho}$ on a set $\boldsymbol{X}$. Show that $\boldsymbol{\rho}$ is reflexive, symmetric and transi-

tive. Hence deduce that $\boldsymbol{\rho}$ is an equivalence relation.

We firstly observe that the leading diagonal (from top left down to bottom right) consists entirely of ones. So $\boldsymbol{\rho}$ is reflexive.

Next, it is easy to see that the matrix is symmetric (as the matrix and its transpose are identical). So $\boldsymbol{\rho}$ is a symmetric relation.

Finally, the Boolean product of $\boldsymbol{M}$ with itself gives a matrix which is identical to $\boldsymbol{M}$. That is, $\boldsymbol{M} \cdot \boldsymbol{M} = \boldsymbol{M}$. This satisfies the requirement that $\boldsymbol{M} \cdot \boldsymbol{M} \leq \boldsymbol{M}$, whence $\boldsymbol{\rho}$ is transitive.

Since $\boldsymbol{\rho}$ is reflexive, symmetric and transitive, it is an equivalence relation.

End of worked example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly decide if a relation is reflexive, symmetric and transitive and hence if it is an equivalence relation.

End of formative assessment

Partitions

Let X be a nonempty set. To *partition* X is to split it up into subsets. But we want the subsets to be disjoint, and we don't want any of them to be the empty set.

Formally, we define a *partition* π of X to be a set of disjoint nonempty subsets of X.

Example:
Let $X = \{1, 2, 3, 4, 5\}$. One partition of X is

$$\pi_1 = \{\{1\}, \{2, 3\}, \{4, 5\}\}.$$

Another is

$$\pi_2 = \{\{1, 3, 5\}, \{2, 4\}\}$$

which gathers together the odd numbers of X in one subset and the even numbers of X in another subset.

These are only two examples of the numerous ways that X can be partitioned.

End of example

Lemma:
Every partition π of a set X determines an equivalence relation ρ_π on X.

End of lemma

What the lemma means is this. We define an equivalence relation ρ_π on X by saying that two elements of X are equiv-

alent if they lie in the same subset of $\boldsymbol{X}$ determined by the partition.

Example:
In the equivalence relation $\boldsymbol{\rho_{\pi_1}}$ determined by the partition $\boldsymbol{\pi_1}$ of the last example, 1 is only equivalent to itself whereas 2 and 3 are equivalent to each other and 4 and 5 are equivalent to each other.

So $\boldsymbol{\rho_{\pi_1}}$ is the set:

$$\begin{aligned}\{(1,1),(2,2),(2,3),(3,2),(3,3),\\ (4,4),(4,5),(5,4),(5,5)\}\end{aligned}$$

The partition $\boldsymbol{\pi_2}$ of the last example determines an equivalence relation $\boldsymbol{\rho_{\pi_2}}$ in which 1, 3 and 5 are all equivalent to each other and in which 2 and 4 are equivalent to each other.

So $\boldsymbol{\rho_{\pi_2}}$ is the set

$$\begin{aligned}\{(1,1),(1,3),(1,5),(2,2),(2,4),\\ (3,1),(3,3),(3,5),(4,2),\\ (4,4),(5,1),(5,3),(5,5)\}\end{aligned}$$

of ordered pairs.

End of example

The next lemma is the converse of the previous one.

Lemma:
If $\boldsymbol{\rho}$ is an equivalence relation on a nonempty set $\boldsymbol{X}$ then $\boldsymbol{\rho}$ determines a par-

tition $\boldsymbol{\pi_\rho}$ of $\boldsymbol{X}$.

End of lemma

The lemma can be applied in this way. We look at any one element of $\boldsymbol{X}$, and then gather together with it all those elements that are $\boldsymbol{\rho}$-equivalent to it. That creates one of the subsets in the partition. Next we look at another element of $\boldsymbol{X}$, but one which isn't $\boldsymbol{\rho}$-equivalent to the first one, and locate all the elements that are $\boldsymbol{\rho}$-equivalent to it; and so on.

If $\boldsymbol{\rho}$ is an equivalence on $\boldsymbol{X}$ and $\boldsymbol{x}$ is an element of $\boldsymbol{X}$, then the set of all elements in $\boldsymbol{X}$ which are $\boldsymbol{\rho}$-equivalent to $\boldsymbol{x}$ is called the *equivalence class* of $\boldsymbol{x}$. The set of distinct equivalence classes is the partition of $\boldsymbol{X}$ *induced by* $\boldsymbol{\rho}$.

Where there is no ambiguity (in an example where we are discussing only one equivalence relation) we denote the equivalence class of $\boldsymbol{x}$ by $[\boldsymbol{x}]$. But where there may be ambiguity we denote it by $\boldsymbol{x\rho}$.

Worked Example:
Let $\boldsymbol{X} = \boldsymbol{\{a, b, c\}}$. Let $\boldsymbol{\rho}$ be the equivalence relation

$$\boldsymbol{\{(a, a), (a, d), (b, b), (c, c),}$$
$$\boldsymbol{(d, a), (d, d)\}.}$$

Find all the distinct equivalence classes of $\boldsymbol{\rho}$, and hence write down the partition $\boldsymbol{\pi_\rho}$ of $\boldsymbol{X}$ induced by $\boldsymbol{\rho}$.

Since $\boldsymbol{a}$ is in an ordered pair with $\boldsymbol{d}$, $\boldsymbol{a}$ and $\boldsymbol{d}$ are equivalent. Since $\boldsymbol{a}$ is in no ordered pairs with anything except itself and $\boldsymbol{d}$, the equivalence class of $\boldsymbol{a}$ just has $\boldsymbol{a}$ and $\boldsymbol{d}$ in it. That is,

$$[\boldsymbol{a}] = \boldsymbol{\{a, d\}}.$$

Now since $\boldsymbol{d}$ is in this class, the class is also the equivalence class of $\boldsymbol{d}$. That is,

$$[\boldsymbol{d}] = \boldsymbol{\{a, d\}} = [\boldsymbol{a}].$$

Next, since $\boldsymbol{b}$ is equivalent only to itself, we have

$$[\boldsymbol{b}] = \boldsymbol{\{b\}}$$

and similarly

$$[\boldsymbol{c}] = \boldsymbol{\{c\}}.$$

Thus the distinct equivalence classes of $\boldsymbol{\rho}$ are $\boldsymbol{\{a, d\}}$, $\boldsymbol{\{b\}}$ and $\boldsymbol{\{c\}}$. The set of distinct equivalence classes is therefore

$$\boldsymbol{\{\{a, d\}, \{b\}, \{c\}\}}$$

which is the partition $\boldsymbol{\pi_\rho}$ of $\boldsymbol{X}$ induced by the equivalence relation $\boldsymbol{\rho}$.

End of worked example

Formative Assessment

You should now do as many practice exercises as necessary to establish that you can correctly find the equivalence relation determined by a partition of a set, and the partition induced by an equivalence relation on a set.

End of formative assessment
